

**THE GENERALIZED FIBONACCI AND LUCAS SOLUTIONS OF  
THE PELL EQUATIONS  $x^2 - (a^2b^2 - b)y^2 = N$  AND  
 $x^2 - (a^2b^2 - 2b)y^2 = N$**

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**ABSTRACT.** In this study, we find continued fraction expansion of  $\sqrt{d}$  when  $d = a^2b^2 - b$  and  $d = a^2b^2 - 2b$  where  $a$  and  $b$  are positive integers. We consider the integer solutions of the Pell equations  $x^2 - (a^2b^2 - b)y^2 = N$  and  $x^2 - (a^2b^2 - 2b)y^2 = N$  when  $N \in \{\pm 1, \pm 4\}$ . We formulate the  $n$ -th solution  $(x_n, y_n)$  by using the continued fraction expansion. We also formulate the  $n$ -th solution  $(x_n, y_n)$  in terms of generalized Fibonacci and Lucas sequences.

### 1. Introduction and Preliminaries

The equation  $x^2 - dy^2 = N$ , with given integers  $d$  and  $N$ , unknowns  $x$  and  $y$ , is called as Pell equation. In the literature, there are several methods for finding the integer solutions of Pell equation such as the Lagrange-Matthews-Mollin algorithm, the cyclic method, Lagrange's system of reductions, use of binary quadratic forms, etc.

If  $d$  is negative, the equation can have only a finite number of solutions. If  $d$  is a perfect square, i.e.  $d = a^2$ , the equation reduces to  $(x - ay)(x + ay) = N$  and there is only a finite number of solutions. If  $d$  is a positive integer but not a perfect square, then simple continued fractions are very useful. The simple continued fraction expansion of  $\sqrt{d}$  has the form  $\sqrt{d} = [a_0, \overline{a_1, a_2, a_3, \dots, a_{m-1}, 2a_0}]$  with  $a_0 = [\sqrt{d}]$ . If the fundamental solution of  $x^2 - dy^2 = 1$  is  $x = x_1$  and  $y = y_1$ , then all nontrivial solutions are given by  $x = x_n$  and  $y = y_n$ , where  $x_n + y_n\sqrt{d} = (x_1 + y_1\sqrt{d})^n$ . If a single solution  $(x, y) = (g, h)$  of the equation  $x^2 - dy^2 = N$  is known, other solutions can be found. Let  $(r, s)$  be a solution of the unit form  $x^2 - dy^2 = 1$ . Then  $(x, y) = (gr \pm dhs, gs \pm hr)$  are solutions of the equation  $x^2 - dy^2 = N$ .

Given a continued fraction expansion of  $\sqrt{d}$ , where all the  $a_i$ 's are real and all except possibly  $a_0$  are positive, define sequences  $\{p_n\}$  and  $\{q_n\}$  by  $p_{-2} = 0, p_{-1} = 1, p_k = a_k p_{k-1} + p_{k-2}$  and  $q_{-2} = 1, q_{-1} = 0, q_k = a_k q_{k-1} + q_{k-2}$  for  $k \geq 0$ . Let  $m$  be the length of the period of continued fraction. Then the fundamental solution of  $x^2 - dy^2 = 1$  is

$$(x_1, y_1) = \begin{cases} (p_{m-1}, q_{m-1}) & \text{if } m \text{ is even} \\ (p_{2m-1}, q_{2m-1}) & \text{if } m \text{ is odd.} \end{cases}$$

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If the length of the period of continued fraction is even, then the equation  $x^2 - dy^2 = -1$  has no integer solutions. If  $m$  is odd, the fundamental solution of  $x^2 - dy^2 = -1$  is given by  $(x_1, y_1) = (p_{m-1}, q_{m-1})$  [3].

Now let us give the following known theorems [4] that will be needed for the next section.

**Theorem 1.** *Let  $d \equiv 2 \pmod{4}$  or  $d \equiv 3 \pmod{4}$ . Then the equation  $x^2 - dy^2 = -4$  has positive integer solutions if and only if the equation  $x^2 - dy^2 = -1$  has positive integer solutions.*

**Theorem 2.** *Let  $d \equiv 0 \pmod{4}$ . If fundamental solution of the equation  $x^2 - (d/4)y^2 = 1$  is  $x_1 + y_1\sqrt{d/4}$ , then fundamental solution of the equation  $x^2 - dy^2 = 4$  is  $(2x_1, y_1)$ .*

**Theorem 3.** *Let  $d \equiv 1 \pmod{4}$  or  $d \equiv 2 \pmod{4}$  or  $d \equiv 3 \pmod{4}$ . If fundamental solution of the equation  $x^2 - dy^2 = 1$  is  $x_1 + y_1\sqrt{d}$ , then fundamental solution of the equation  $x^2 - dy^2 = 4$  is  $(2x_1, 2y_1)$ .*

In this study [2], since generalized Fibonacci and Lucas sequences related solutions of the forthcoming Pell equations are going to be taken into consideration, let us briefly recall the generalized Fibonacci sequences  $\{U_n(k, s)\}$  and Lucas sequences  $\{V_n(k, s)\}$ . Let  $k$  and  $s$  be two non-zero integers with  $k^2 + 4s > 0$ . Generalized Fibonacci sequence is defined by

$$U_0(k, s) = 0, U_1(k, s) = 1$$

and

$$U_{n+1}(k, s) = kU_n(k, s) + sU_{n-1}(k, s)$$

for  $n \geq 1$ . Generalized Lucas sequence is defined by

$$V_0(k, s) = 2, V_1(k, s) = k$$

and

$$V_{n+1}(k, s) = kV_n(k, s) + sV_{n-1}(k, s)$$

for  $n \geq 1$ . It is also well-known from the literature that generalized Fibonacci and Lucas numbers have many interesting and significant properties. Binet's formulas are probably the most important one among them. For generalized Fibonacci and Lucas sequences, Binet's formulas are given by  $U_n(k, s) = \frac{\alpha^n - \beta^n}{\alpha - \beta}$  and  $V_n(k, s) = \alpha^n + \beta^n$  where  $\alpha = (k + \sqrt{k^2 + 4s})/2$  and  $\beta = (k - \sqrt{k^2 + 4s})/2$  [5].

There are a large number of studies concerning Pell equation in the literature. Güney [1] solved the Pell equations  $x^2 - (a^2b^2 + 2b)y^2 = N$  when  $N \in \{\pm 1, \pm 4\}$ .

In this study, we consider the integer solutions of the Pell equations

$$x^2 - (a^2b^2 - b)y^2 = N \text{ and } x^2 - (a^2b^2 - 2b)y^2 = N$$

in terms of the generalized Fibonacci and Lucas numbers.

## 2. The Pell equation $x^2 - (a^2b^2 - b)y^2 = N$

Firstly, we consider the integer solutions of the Pell equation

$$(1) \quad E : x^2 - (a^2b^2 - b)y^2 = 1.$$

$$x^2 - (a^2b^2 - b)y^2 = N \text{ AND } x^2 - (a^2b^2 - 2b)y^2 = N$$

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**Theorem 4.** Let  $E$  be the Pell equation in (1) and  $a \geq 2$ . Then the followings hold:

(i) The continued fraction expansion of  $\sqrt{a^2b^2 - b}$  is

$$\sqrt{a^2b^2 - b} = \begin{cases} [a-1; \overline{1, 2a-2}] & \text{if } b = 1 \\ [ab-1; \overline{1, 2a-2, 1, 2ab-2}] & \text{if } b > 1. \end{cases}$$

(ii) The fundamental solution is

$$(x_1, y_1) = \begin{cases} (a, 1) & \text{if } b = 1 \\ (2a^2b - 1, 2a) & \text{if } b > 1. \end{cases}$$

(iii) The  $n$ -th solution  $(x_n, y_n)$  can be find by

$$\frac{x_n}{y_n} = \begin{cases} [a-1; (1, 2a-2)_{n-1}, 1] & \text{if } b = 1 \\ [ab-1; (1, 2a-2, 1, 2ab-2)_{n-1}, 1] & \text{if } b > 1. \end{cases}$$

where  $(1, 2a-2)_{n-1}$  and  $(1, 2a-2, 1, 2ab-2)_{n-1}$  mean that there are  $n-1$  successive terms  $(1, 2a-2)$  and  $(1, 2a-2, 1, 2ab-2)$ , respectively.

*Proof.* (i) If  $b = 1$ , then

$$\begin{aligned} \sqrt{a^2 - 1} &= a - 1 + (\sqrt{a^2 - 1} - (a - 1)) = a - 1 + \frac{2a-2}{\sqrt{a^2-1}+a-1} = a - 1 + \frac{1}{\frac{\sqrt{a^2-1}+a-1}{2a-2}} \\ &= a - 1 + \frac{1}{1 + \frac{\sqrt{a^2-1}-a+1}{2a-2}} = a - 1 + \frac{1}{1 + \frac{1}{\sqrt{a^2-1}+a-1}} \\ &= a - 1 + \frac{1}{1 + \frac{1}{\sqrt{a^2-1}+a-1+a-1-(a-1)}} = a - 1 + \frac{1}{1 + \frac{1}{2a-2+(\sqrt{a^2-1}-(a-1))}}. \end{aligned}$$

Therefore  $\sqrt{a^2 - 1} = [a-1; \overline{1, 2a-2}]$ .

If  $b > 1$ , then

$$\begin{aligned} \sqrt{a^2b^2 - b} &= ab - 1 + (\sqrt{a^2b^2 - b} - (ab - 1)) \\ &= ab - 1 + \frac{1}{\frac{\sqrt{a^2b^2 - b} + (ab-1)}{2ab-b-1}} = ab - 1 + \frac{1}{1 + \frac{\sqrt{a^2b^2 - b} - (ab-b)}{2ab-b-1}} \\ &= ab - 1 + \frac{1}{1 + \frac{1}{\frac{\sqrt{a^2b^2 - b} + (ab-b)}{b}}} = ab - 1 + \frac{1}{1 + \frac{1}{2a-2 + \frac{\sqrt{a^2b^2 - b} - (ab-b)}{b}}} \\ &= ab - 1 + \frac{1}{1 + \frac{1}{2a-2 + \frac{1}{\frac{\sqrt{a^2b^2 - b} + (ab-b)}{2ab-b-1}}}} = ab - 1 + \frac{1}{1 + \frac{1}{2a-2 + \frac{1}{1 + \frac{1}{\sqrt{a^2b^2 - b} + (ab-b)}}}} \\ &= ab - 1 + \frac{1}{1 + \frac{1}{2a-2 + \frac{1}{1 + \frac{1}{\frac{2ab-b-1}{\sqrt{a^2b^2 - b} - (ab-1)}}}}} = ab - 1 + \frac{1}{1 + \frac{1}{2a-2 + \frac{1}{1 + \frac{1}{\sqrt{a^2b^2 - b} + (ab-1)}}}} \\ &= ab - 1 + \frac{1}{1 + \frac{1}{2a-2 + \frac{1}{1 + \frac{1}{2ab-2 + (\sqrt{a^2b^2 - b} - (ab-1))}}}}. \end{aligned}$$

This completes the proof.

(ii) If  $b = 1$ , then from  $x_n = p_{nm-1}$  and  $y_n = q_{nm-1}$ , we obtain  $x_1 = p_1$  and  $y_1 = q_1$ . Therefore we must find  $p_1$  and  $q_1$ . It is easily seen that  $p_1 = a_1p_0 + p_{-1} = a$  and  $q_1 = a_1q_0 + q_{-1} = 1$ . That is, the fundamental solution of  $x^2 - (a^2 - 1)y^2 = 1$  is  $(x_1, y_1) = (a, 1)$ .

If  $b > 1$ , then from  $x_n = p_{nm-1}$  and  $y_n = q_{nm-1}$ , we obtain  $x_1 = p_3$  and  $y_1 = q_3$ . Therefore we must find  $p_3$  and  $q_3$ . Now, we can find in a different way with the help of 3th convergent of  $\sqrt{a^2b^2 - b}$ .

$\frac{p_3}{q_3} = [ab - 1; 1, 2a - 2, 1] = ab - 1 + \frac{1}{1 + \frac{1}{2a - 2 + \frac{1}{1}}} = \frac{2a^2b - 1}{2a}$ . Therefore  $(x_1, y_1) = (2a^2b - 1, 2a)$ .

(iii) If  $b = 1$ , then it is known that  $(x_1, y_1) = (a, 1)$ . For  $n = 1$ , we obtain  $\frac{x_1}{y_1} = [a - 1; 1] = a - 1 + \frac{1}{1} = \frac{a}{1}$ . Hence it is true for  $n = 1$ .

We assume that  $(x_n, y_n)$  is a solution of  $x^2 - (a^2 - 1)y^2 = 1$ . That is,  $\frac{x_n}{y_n} = [a - 1; (1, 2a - 2)_{n-1}, 1]$ .

Now we must show that it holds for  $(x_{n+1}, y_{n+1})$ .

$$\begin{aligned} \frac{x_{n+1}}{y_{n+1}} &= a - 1 + \frac{1}{1 + \frac{1}{2a - 2 + \frac{1}{1 + \frac{1}{2a - 2 + \frac{1}{\dots 2a - 2 + 1}}}}} \\ &= a - 1 + \frac{1}{1 + \frac{1}{a - 1 + a - 1 + \frac{1}{1 + \frac{1}{2a - 2 + \frac{1}{\dots 2a - 2 + 1}}}}} \\ &= a - 1 + \frac{1}{1 + \frac{1}{a - 1 + \frac{x_n}{y_n}}} \\ &= \frac{ax_n + (a^2 - 1)y_n}{ay_n + x_n}. \end{aligned}$$

$(x_{n+1}, y_{n+1})$  is a solution of  $x^2 - (a^2 - 1)y^2 = 1$  since  $x_{n+1}^2 - (a^2 - 1)y_{n+1}^2 = (ax_n + (a^2 - 1)y_n)^2 - (a^2 - 1)(ay_n + x_n)^2 = x_n^2 - (a^2 - 1)y_n^2 = 1$ .

If  $b > 1$ , then the proof is made by induction in a similar way.  $\square$

**Theorem 5.** All positive integer solutions of the equation  $x^2 - (a^2b^2 - b)y^2 = 1$  are given by

$$(x_n, y_n) = \begin{cases} ((V_n(2a, -1))/2, U_n(2a, -1)) & \text{if } b = 1 \\ ((V_n(4a^2b - 2, -1))/2, 2aU_n(4a^2b - 2, -1)) & \text{if } b > 1 \end{cases}$$

with  $n \geq 1$ .

*Proof.* If  $b > 1$ , then the followings hold:

By Theorem 4-ii, all positive integer solutions of the equation  $x^2 - (a^2b^2 - b)y^2 = 1$  are given by

$$x_n + y_n\sqrt{a^2b^2 - b} = \left(2a^2b - 1 + 2a\sqrt{a^2b^2 - b}\right)^n$$

with  $n \geq 1$ . Assume that  $\alpha = 2a^2b - 1 + 2a\sqrt{a^2b^2 - b}$  and  $\beta = 2a^2b - 1 - 2a\sqrt{a^2b^2 - b}$ . Then  $\alpha - \beta = 4a\sqrt{a^2b^2 - b}$ .

$$x_n + y_n\sqrt{a^2b^2 - b} = \alpha^n$$

and

$$x_n - y_n\sqrt{a^2b^2 - b} = \beta^n.$$

Therefore  $x_n = \frac{\alpha^n + \beta^n}{2} = \frac{V_n(4a^2b - 2, -1)}{2}$  and  $y_n = \frac{\alpha^n - \beta^n}{2\sqrt{a^2b^2 - b}} = 2a\frac{\alpha^n - \beta^n}{\alpha - \beta} = 2aU_n(4a^2b - 2, -1)$ . That is,  $(x_n, y_n) = \left(\frac{V_n(4a^2b - 2, -1)}{2}, 2aU_n(4a^2b - 2, -1)\right)$ .

$$x^2 - (a^2b^2 - b)y^2 = N \text{ AND } x^2 - (a^2b^2 - 2b)y^2 = N \quad 5$$

Similarly it can be shown that if  $b = 1$ , then  $(x_n, y_n) = \left( \frac{V_n(2a, -1)}{2}, U_n(2a, -1) \right)$ .  $\square$

**Theorem 6.** *The Pell equation  $x^2 - (a^2b^2 - b)y^2 = -1$  has no positive integer solutions.*

*Proof.* The lengths of the period of continued fraction  $\sqrt{a^2b^2 - b}$  are even, then this equation has no positive integer solutions.  $\square$

**Theorem 7.** *The fundamental solution of the Pell equation  $x^2 - (a^2b^2 - b)y^2 = 4$  is*

$$(x_1, y_1) = \begin{cases} (2a, 2) & \text{if } b = 1 \\ (4a^2b - 2, 4a) & \text{if } b > 1. \end{cases}$$

*Proof.* It is obvious from Theorem 3 and Theorem 4-ii.  $\square$

**Theorem 8.** *All positive integer solutions of the equation  $x^2 - (a^2b^2 - b)y^2 = 4$  are given by*

$$(x_n, y_n) = \begin{cases} (V_n(2a, -1), 2U_n(2a, -1)) & \text{if } b = 1 \\ (V_n(4a^2b - 2, -1), 4aU_n(4a^2b - 2, -1)) & \text{if } b > 1 \end{cases}$$

with  $n \geq 1$ .

*Proof.* It is trivial from Theorem 3 and Theorem 5.  $\square$

**Corollary 1.** *All positive integer solutions of the equation  $x^2 - (9k^2 - 3)y^2 = 1$  are given by*

$$(x_n, y_n) = ((V_n(12k^2 - 2, -1)) / 2, 2kU_n(12k^2 - 2, -1))$$

with  $n \geq 1$ .

**Corollary 2.** *All positive integer solutions of the equation  $x^2 - (9k^2 - 3)y^2 = 4$  are given by*

$$(x_n, y_n) = (V_n(12k^2 - 2, -1), 4kU_n(12k^2 - 2, -1))$$

with  $n \geq 1$ .

### 3. The Pell equation $x^2 - (a^2b^2 - 2b)y^2 = N$

Now, we consider the integer solutions of the Pell equation

$$(2) \quad F : x^2 - (a^2b^2 - 2b)y^2 = 1.$$

**Theorem 9.** *Let  $F$  be the Pell equation in (2) and  $a \geq 3$ . Then the followings hold:*

(i) *The continued fraction expansion of  $\sqrt{a^2b^2 - 2b}$  is*

$$\sqrt{a^2b^2 - 2b} = [ab - 1; 1, a - 2, 1, 2ab - 2].$$

(ii) *The fundamental solution is*

$$(x_1, y_1) = (a^2b - 1, a).$$

(iii) *The  $n$ -th solution  $(x_n, y_n)$  can be find by*

$$\frac{x_n}{y_n} = [ab - 1; (1, a - 2, 1, 2ab - 2)_{n-1}, 1]$$

where  $(1, a - 2, 1, 2ab - 2)_{n-1}$  means that there are  $n-1$  successive terms  $(1, a - 2, 1, 2ab - 2)$ .

*Proof.* (i)

$$\begin{aligned}
\sqrt{a^2b^2 - 2b} &= ab - 1 + (\sqrt{a^2b^2 - 2b} - (ab - 1)) \\
&= ab - 1 + \frac{1}{\frac{\sqrt{a^2b^2 - 2b} + (ab - 1)}{2ab - 2b - 1}} = ab - 1 + \frac{1}{1 + \frac{\sqrt{a^2b^2 - 2b} - (ab - 2b)}{2ab - 2b - 1}} \\
&= ab - 1 + \frac{1}{1 + \frac{1}{\frac{\sqrt{a^2b^2 - 2b} + (ab - 2b)}{2b}}} = ab - 1 + \frac{1}{1 + \frac{1}{a - 2 + \frac{\sqrt{a^2b^2 - 2b} - (ab - 2b)}{2b}}} \\
&= ab - 1 + \frac{1}{1 + \frac{1}{a - 2 + \frac{1}{\frac{\sqrt{a^2b^2 - 2b} + (ab - 2b)}{2ab - 2b - 1}}}} = ab - 1 + \frac{1}{1 + \frac{1}{a - 2 + \frac{1}{1 + \frac{1}{2ab - 2b - 1}}}}} \\
&= ab - 1 + \frac{1}{1 + \frac{1}{a - 2 + \frac{1}{1 + \frac{1}{\sqrt{a^2b^2 - 2b} - (ab - 1)}}}} = ab - 1 + \frac{1}{1 + \frac{1}{a - 2 + \frac{1}{1 + \frac{1}{2ab - 2 + (\sqrt{a^2b^2 - 2b} - (ab - 1))}}}}.
\end{aligned}$$

This completes the proof.

(ii) From  $x_n = p_{nm-1}$  and  $y_n = q_{nm-1}$ , we obtain  $x_1 = p_3$  and  $y_1 = q_3$ . Therefore we must find  $p_3$  and  $q_3$ . Now, we can find with the help of 3th convergent of  $\sqrt{a^2b^2 - 2b}$ .

$\frac{p_3}{q_3} = [ab - 1; 1, a - 2, 1] = ab - 1 + \frac{1}{1 + \frac{1}{a - 2 + \frac{1}{1}}} = \frac{a^2b - 1}{a}$ . Therefore  $(x_1, y_1) = (a^2b - 1, a)$ .

(iii) The proof is made by a similar manner as in the proof of the Theorem 4-iii.  $\square$

**Theorem 10.** All positive integer solutions of the equation  $x^2 - (a^2b^2 - 2b)y^2 = 1$  are given by

$$(x_n, y_n) = ((V_n(2a^2b - 2, -1)) / 2, aU_n(2a^2b - 2, -1))$$

with  $n \geq 1$ .

*Proof.* By Theorem 9-ii, all positive integer solutions of the equation  $x^2 - (a^2b^2 - 2b)y^2 = 1$  are given by

$$x_n + y_n \sqrt{a^2b^2 - 2b} = (a^2b - 1 + a\sqrt{a^2b^2 - 2b})^n$$

with  $n \geq 1$ . Assume that  $\alpha = a^2b - 1 + a\sqrt{a^2b^2 - 2b}$  and  $\beta = a^2b - 1 - a\sqrt{a^2b^2 - 2b}$ . Then  $\alpha - \beta = 2a\sqrt{a^2b^2 - 2b}$ .

$$x_n + y_n \sqrt{a^2b^2 - 2b} = \alpha^n$$

and

$$x_n - y_n \sqrt{a^2b^2 - 2b} = \beta^n.$$

Therefore  $x_n = \frac{\alpha^n + \beta^n}{2} = \frac{V_n(2a^2b - 2, -1)}{2}$  and  $y_n = \frac{\alpha^n - \beta^n}{2\sqrt{a^2b^2 - 2b}} = a \frac{\alpha^n - \beta^n}{\alpha - \beta} = aU_n(2a^2b - 2, -1)$ . That is,  $(x_n, y_n) = \left( \frac{V_n(2a^2b - 2, -1)}{2}, aU_n(2a^2b - 2, -1) \right)$ .  $\square$

**Theorem 11.** The Pell equation  $x^2 - (a^2b^2 - 2b)y^2 = -1$  has no positive integer solutions.

*Proof.* The lengths of the period of continued fraction  $\sqrt{a^2b^2 - 2b}$  is even, then this equation has no positive integer solutions.  $\square$

$$x^2 - (a^2b^2 - b)y^2 = N \text{ AND } x^2 - (a^2b^2 - 2b)y^2 = N \quad 7$$

**Theorem 12.** *The fundamental solution of the Pell equation  $x^2 - (a^2b^2 - 2b)y^2 = 4$  is*

$$(x_1, y_1) = (2a^2b - 2, 2a).$$

*Proof.* It is clear from Theorem 3 and Theorem 9-ii. □

**Theorem 13.** *All positive integer solutions of the equation  $x^2 - (a^2b^2 - 2b)y^2 = 4$  are given by*

$$(x_n, y_n) = (V_n(2a^2b - 2, -1), 2aU_n(2a^2b - 2, -1))$$

*with  $n \geq 1$ .*

*Proof.* The proof can be easily seen from Theorem 3 and Theorem 10. □

**Theorem 14.** *The Pell equation  $x^2 - (a^2b^2 - 2b)y^2 = -4$  has no positive integer solutions.*

*Proof.* Let  $b$  be odd. If  $a$  is odd, then  $a^2b^2 - 2b \equiv 3 \pmod{4}$ . If  $a$  is even, then  $a^2b^2 - 2b \equiv 2 \pmod{4}$ . From Theorem 1, we know that the equation  $x^2 - (a^2b^2 - 2b)y^2 = -4$  has positive integer solutions if and only if the equation  $x^2 - (a^2b^2 - 2b)y^2 = -1$  has positive integer solutions. But from Theorem 11, the equation  $x^2 - (a^2b^2 - 2b)y^2 = -1$  has no positive integer solutions. Therefore, the equation  $x^2 - (a^2b^2 - 2b)y^2 = -4$  has no positive integer solutions.

Let  $b$  be even. Then  $a^2b^2 - 2b$  is even. Assume by way of contradiction that there are positive integers  $m$  and  $n$  such that  $m^2 - (a^2b^2 - 2b)n^2 = -4$ . Both  $b$  and  $a^2b^2 - 2b$  are even. Therefore,  $m$  is even. Let  $b = 2t$  where  $t \in \mathbb{Z}^+$ . Then  $m^2 - (a^24t^2 - 4t)n^2 = -4$  and we get  $(m/2)^2 - (a^2t^2 - t)n^2 = -1$ . We know from Theorem 6, the equation  $x^2 - (a^2b^2 - b)y^2 = -1$  has no positive integer solutions. So this is a contradiction. Then the Pell equation  $x^2 - (a^2b^2 - 2b)y^2 = -4$  has no positive integer solutions. □

**Corollary 3.** *All positive integer solutions of the equation  $x^2 - (9k^2 - 6)y^2 = 1$  are given by*

$$(x_n, y_n) = ((V_n(6k^2 - 2, -1))/2, kU_n(6k^2 - 2, -1))$$

*with  $n \geq 1$ .*

**Corollary 4.** *All positive integer solutions of the equation  $x^2 - (9k^2 - 6)y^2 = 4$  are given by*

$$(x_n, y_n) = (V_n(6k^2 - 2, -1), 2kU_n(6k^2 - 2, -1))$$

*with  $n \geq 1$ .*

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